18/4/23 MATH2060 Lecture

9.4 Series of functions

- Sequences of fn's $f_n \to f$.
  - uniform convergence $\Rightarrow$ f_n cts, then f is cts. ...
    $f_n \in R[a,b]$, then $f \in R[a,b]$.
- Infinite Series $\sum_{n=1}^{\infty} a_n$, $a_n \in \mathbb{R}$.
  - Root Test, Ratio Test, Comparison Thms, Absolute Convergence, ...

Want to talk about infinite series of functions.

$$f_1(x) + f_2(x) + \cdots = \sum_{n=1}^{\infty} f_n(x)$$

Power Series: $\sum_{n=0}^{\infty} a_n (x-c)^n$.

Def (9.4.1): let (f_n) be a sequence of functions, $f_n : D \subset \mathbb{R} \to \mathbb{R}$, then the sequence of partial sums (S_n) of the infinite series of functions $\sum_{n=1}^{\infty} f_n$, is defined by $S_n(x) := \sum_{k=1}^{n} f_k(x)$, $\forall x \in D$.

- If (s_n) converges to a function f on D, then we say that the infinite series $\sum_{n=1}^{\infty} f_n$ converges to f on D.

$$\left(\forall x \in D,\ \forall \varepsilon > 0,\ \exists K(\varepsilon, x) \in \mathbb{N},\ \text{s.t. if } n \geq K, \text{ then } |s_n(x) - f(x)| < \varepsilon. \right)$$

We write $f(x) = \sum_{n=1}^{\infty} f_n(x)$, $f = \sum f_n$.

- If $\sum_{n=1}^{\infty} |f_n(x)|$ converges $\forall x \in D$, then we say $\sum_{n=1}^{\infty} f_n$ is absolutely convergent

$(s_n \rightrightarrows f)$
- If s_n converges uniformly to f on D, then we say $\sum_{n=1}^{\infty} f_n$ is uniformly convergent to f on D.

$\forall \varepsilon > 0$, $\exists K = K(\varepsilon) \in \mathbb{N}$ s.t. if $n \geq K$, then $|s_n(x) - f(x)| < \varepsilon$, $\forall x \in D$.

Thm 9.4.2: If each f_n is continuous on D, $\forall n \in \mathbb{N}$, and $\sum f_n$ converges uniformly to f on D, then f is continuous on D.

Thm 9.4.3: If each $f_n \in R[a,b]$, $\forall n \in \mathbb{N}$, and $\sum f_n$ converges uniformly to f on D, then $f \in R[a,b]$ and $\int_a^b f = \sum_{n=1}^{\infty} \int_a^b f_n$.

(Interchanging the limit:
$$\int_a^b f = \int_a^b \sum_{n=1}^{\infty} f_n = \sum_{n=1}^{\infty} \int_a^b f_n .$$
)

Thm 9.4.4 If each $f_n : [a,b] \to \mathbb{R}$, $n \in \mathbb{N}$,

- f_n' exists on $[a,b]$, $\forall n \in \mathbb{N}$
- $\exists x_0 \in [a,b]$ s.t. $\sum f_n(x_0)$ converges.
- $\sum_{n=1}^{\infty} f_n'$ converges uniformly on $[a,b]$

Then $\exists f:[a,b]\to\mathbb{R}$ s.t.

- $\sum_{n=1}^{\infty} f_n$ converge uniformly to f on $[a,b]$
- f' exists and $f' = \sum_{n=1}^{\infty} f_n'$

Pf (of Thm 9.4.2 – 9.4.4): Apply corresponding result for sequences of functions to the partial sums (s_n).

Tests for Uniform Convergence

Thm 9.4.5 (Cauchy Criteria): $\sum_{n=1}^{\infty} f_n$ is uniformly convergent on D if and only if $\forall \varepsilon>0,\ \exists K(\varepsilon)\in\mathbb{N}$ s.t.

$$\text{if } m>n\geq K(\varepsilon),\ |f_{n+1}(x)+\dots+f_m(x)|<\varepsilon,\ \forall x\in D$$

Pf: Apply Cauchy Criteria for uniform convergence for sequences of functions to (s_n) (Thm 8.1.10): $s_n \rightrightarrows f$ iff $\forall \varepsilon > 0$ $\exists K(\varepsilon) \in \mathbb{N}$ s.t.

if $m > n \geq K$, $|s_m(x) - s_n(x)| < \varepsilon$.

$$\Updownarrow$$

$$|f_{n+1}(x) + \dots + f_m(x)| < \varepsilon.$$

<u>Thm 9.4.6</u> (Weierstrass M-test)

If 1) $|f_n(x)| \leq M_n$ $\forall n \in \mathbb{N}$, $x \in D$

2) $\sum_{n=1}^{\infty} M_n$ is convergent (not necessarily uniformly convergent)

Then $\sum_{n=1}^{\infty} f_n$ is uniformly convergent.

Pf: $0 \leq M_n$ and $\sum_{n=1}^{\infty} M_n$ is convergent implies it is Cauchy ie.

Let $L_n = \sum_{k=1}^{n} M_k$, then $\forall \varepsilon > 0$ $\exists K(\varepsilon) \in \mathbb{N}$, s.t. if $m > n \geq K(\varepsilon)$,

$L_m - L_n < \varepsilon$. (since each $M_n \geq 0$, $L_m \geq L_n$).

$\Updownarrow$

$M_{n+1} + \dots + M_m < \varepsilon$.

By $|f_n(x)| \leq M_n$, we then have

triangle ineq.

$|f_{n+1}(x) + \dots + f_m(x)| \leq M_{n+1} + \dots + M_m < \varepsilon$.

So by the Cauchy Criteria, $\sum_{n=1}^{\infty} f_n$ converges uniformly on D. //

Power Series

Def 9.4.7 $\sum_{n=1}^{\infty} f_n$ is a power series centred at $c \in \mathbb{R}$

if each $f_n(x) = a_n(x-c)^n$. "infinite polynomial" $\sum_{n=0}^{\infty} a_n(x-c)^n$

Rmk: 1) For simplicity, from now on we will only talk about the case where the centre $c=0$. We are free to do this because setting $y = x-c$ (translation), turns any power series centred at c to a power series centred at 0.

2) Index from $n=0$. $\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \dots$

3) $\sum_{n=0}^{\infty} a_n x^n$ may not be defined on all of $\mathbb{R}$.

i) $\sum_{n=0}^{\infty} n! x^n$ converges only at $x=0$. (Exercise, Ratio test).

ii) $\sum_{n=0}^{\infty} x^n$ converges only for $|x|<1$. (Geometric series).

iii) $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ Converges for all $x \in \mathbb{R}$ (Power series expansion for e^x).

So we need to determine the set on which $\sum_{n=0}^{\infty} a_n x^n$ converges.

<u>Recall</u> (Def 3.4.10 & Thm 3.4.11): For (x_n) a bounded sequence, the limit superior of (x_n), is defined as:

$$\limsup(x_n) := \inf\{v \in \mathbb{R} : v < x_n \text{ for finitely many } n\}$$
$$= \inf\{v \in \mathbb{R} : x_n \leq v \text{ for sufficiently large } n\}.$$
$$\exists K(v) \in \mathbb{N} \text{ s.t. if } n \geq K(v), \text{ then } x_n \leq v.$$

i) If $v > \limsup(x_n)$, then
$x_n \leq v$ for sufficiently large n.

ii) If $w < \limsup(x_n)$, then $\exists$ infinitely many $n \in \mathbb{N}$ s.t. $w \leq x_n$.

<u>Def 9.4.8</u>: Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series and

$$\rho = \begin{cases} \limsup(|a_n|^{\frac{1}{n}}) & \text{if } (|a_n|^{\frac{1}{n}}) \text{ is a bounded sequence} \\ +\infty & \text{otherwise.} \end{cases}$$

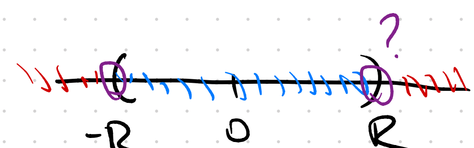
Then the radius of convergence of $\sum_{n=0}^{\infty} a_n x^n$ is

$$R = \frac{1}{\rho} = \begin{cases} 0 & \text{if } \rho = +\infty \\ \dfrac{1}{\limsup \left(|a_n|^{\frac{1}{n}}\right)} & \end{cases} \quad (\text{includes } R = +\infty \text{ if } \rho = 0).$$

Then the interval of convergence is the open interval $(-R, R)$.

Thm 9.4.9 (Cauchy-Hadamard Thm) : If R is the radius of convergence of $\sum_{n=0}^{\infty} a_n x^n$, then $\sum_{n=0}^{\infty} a_n x^n$ is
$$\begin{cases} \text{absolutely convergent} & \text{if } |x| < R \\ \text{divergent} & \text{if } |x| > R. \end{cases}$$

Rmk : No conclusion for $|x| = R$



i) $\sum_{n=0}^{\infty} x^n$: $\rho = \limsup 1 = 1. \Rightarrow R = 1.$

$x=1$: $\sum_{n=0}^{\infty} x^n = 1+1+\dots$ is divergent.

$x=-1$ $\sum_{n=0}^{\infty} x^n = -1+1-1+\dots$ is divergent.

ii) $\sum_{n=0}^{\infty} \frac{1}{n} x^n$: $R=1$. (Exercise)

$x=1$: $\sum_{n=0}^{\infty} \frac{1}{n} x^n = 1+\frac{1}{2}+\frac{1}{3}+\dots$ is divergent

$x=-1$ $\sum_{n=0}^{\infty} \frac{1}{n} x^n = -1+\frac{1}{2}-\frac{1}{3}+\dots$ is convergent.

iii) $\sum_{n=0}^{\infty} \frac{1}{n^2} x^n$, $R=1$, convergent at both $x=1,-1$.

Pf of Cauchy-Hadamard:

Case: $R=0, +\infty$ left as an exercise.

Assume $0<R<+\infty$. Clearly at $x=0$, $\sum_{n=0}^{\infty} a_n x^n$ converges.

Suppose $|x| < R$. Then $\exists\ 0 < c < 1$ such that $|x| = cR = \frac{c}{\rho}$.

$c = \rho|x| = \limsup(|a_n|^{\frac{1}{n}})|x| = \limsup(|a_n|^{\frac{1}{n}}|x|)$

$\exists K(x, c) \in \mathbb{N}$ s.t. if $n \geq K$ $|a_n|^{\frac{1}{n}}|x| \leq c$.

$\Rightarrow |a_n x^n| \leq c^n$, $\forall n \geq K$.

Since $0 < c < 1$, $\sum_{n=0}^{\infty} c^n$ is convergent, so by comparison test (3.7.7) $\sum_{n=0}^{\infty} |a_n x^n|$ converges, so $\sum_{n=0}^{\infty} a_n x^n$ is absolutely convergent.

For $|x| > R$, $|x| > \frac{1}{\rho} \Leftrightarrow \rho > \frac{1}{|x|} \Rightarrow \limsup(|a_n|^{\frac{1}{n}}) > \frac{1}{|x|}$.

$\Rightarrow |a_n|^{\frac{1}{n}} > \frac{1}{|x|}$ for infinitely many n.

$\Rightarrow |a_n x^n| > 1$ for infinitely many n.

$\Rightarrow a_n x^n \not\to 0$, which implies $\sum_{n=0}^{\infty} a_n x^n$ diverges.

Rmks: 1) If $\lim \left|\frac{a_n}{a_{n+1}}\right|$ exists, then $R = \lim \left|\frac{a_n}{a_{n+1}}\right|$.

- Reciprocal of Ratio Test.
- Includes $\lim \left|\frac{a_n}{a_{n+1}}\right| \to +\infty$.

i) $\sum x^n$, $\sum \frac{1}{n} x^n$, $\sum \frac{1}{n^2} x^n$

$$\downarrow$$

$$\left|\frac{a_n}{a_{n+1}}\right| = \frac{1}{n^2} \Big/ \frac{1}{(n+1)^2} = \left(\frac{n+1}{n}\right)^2 \to 1 \text{ as } n \to \infty.$$

$$\Rightarrow R = 1.$$

2) If we can choose $0 < c < 1$ independent of $|x|$, then we would have uniform convergence.

Thm 9.4.10 Let R be the radius of convergence of $\sum_{n=0}^{\infty} a_n x^n$ and $[a,b] \subset (-R,R)$ then $\sum_{n=0}^{\infty} a_n x^n$ is uniformly convergent on $[a,b]$. (closed and bounded)

Rmk: 1) $R=0$ is excluded $(-R, R) = (0,0) = \emptyset$.

2) $(-R,R)$ could include $(-\infty,\infty)$, hence why we need $[a,b]$ closed and bounded.

Pf: Since $[a,b] \subset (-R,R)$ is closed and bounded, $\exists\, 0<c<1$ s.t.

$-cR < a$ and $b < cR$

So $\forall x \in [a,b]$, $|x| < cR$. but now c is independent of $|x|$ (only depends on a,b).

By argument above, $\exists\, K = K(a,b) \in \mathbb{N}$ s.t. $|a_n x^n| \le c^n$, $\forall n \ge K$.

$\sum_{n=0}^{\infty} c^n$ is convergent, so by Weierstrass M-test $\sum_{n=0}^{\infty} a_n x^n$ converges uniformly //

Thm 9.4.11: 1) The limit of a power series is continuous in $(-R,R)$

2) a power series can be integrated term by term

$\left(\int_a^b \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \int_a^b a_n x^n \right)$

over any closed bounded interval $[a,b] \subset (-R,R)$.

Pf: 1) $\forall x \in (-R,R)$, take a,b such that $x \in [a,b] \subset (-R,R)$

Then on $[a,b]$, $\sum_{n=0}^{\infty} a_n x^n$ converges uniformly and hence is continuous on $[a,b]$, in particular at x.

2) Similar. //

Thm 9.4.12 (Differentiation Thm)

A power series can be differentiated term by term within the interval of convergence $(-R,R)$, moreover if R = radius of convergence for

$$f(x) = \sum_{n=0}^{\infty} a_n x^n,$$ then

the radius of convergence of $\sum_{n=0}^{\infty} n a_n x^{n-1}$ is also R and

$$f'(x) = \left(\sum_{n=0}^{\infty} a_n x^n\right)' = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad \text{for } |x|<R.$$

Rmk: This is stronger than the general case because we don't require uniform convergence.

Pf: Since $n^{\frac{1}{n}} \to 1$, the sequence $|na_n|^{\frac{1}{n}}$ is bounded if and only if $|a_n|^{\frac{1}{n}}$ is bounded and moreover

$$\limsup |na_n|^{\frac{1}{n}} = \limsup \left(n^{\frac{1}{n}} |a_n|^{\frac{1}{n}}\right) = \limsup \left(|a_n|^{\frac{1}{n}}\right)$$

So radius of convergence of $\sum_{n=0}^{\infty} n a_n x^{n-1}$ = radius of convergence of $\sum_{n=0}^{\infty} a_n x^n$

$= R$.

$\forall x \in (-R, R)$, choose $0 < a < R$ s.t. $|x| < a$.

Then consider $[-a, a] \subseteq (-R, R)$ is closed bounded

$\cdot$ $0 \in [-a, a]$ s.t. $\sum_{n=0}^{\infty} a_n x^n$ converges at $x = 0$

So by Thm 9.4.10, Thm 8.2.3 and $(a_n x^n)' = n a_n x^{n-1}$

$\sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=1}^{\infty} (a_n x^n)'$ converges uniformly on $[-a, a]$.

So by Thm 9.4.4. $\left(\sum_{n=0}^{\infty} a_n x^n\right)' = \sum_{n=1}^{\infty} n a_n x^{n-1}$ on $[-a, a]$.

Since x was arbitrary, we have the result on $(-R, R)$ //.

Rmk: 1) Thm 9.4.12 makes no assertion about $|x| = R$

e.g. $\sum \frac{1}{n^2} x^n$ converges for $|x| = 1$

but $\left(\sum \frac{1}{n^2} x^n\right)' = \sum \frac{1}{n} x^{n-1}$ converges at $x = -1$

diverges at $x = 1$.

2) Repeated application of Thm 9.4.12 gives

$\forall k \in \mathbb{N}$, $\left(\sum_{n=0}^{\infty} a_n x^n\right)^{(k)} = \sum_{n=k}^{\infty} \frac{n!}{(n-k)!} a_n x^{n-k}$, w/ radius of convergence R